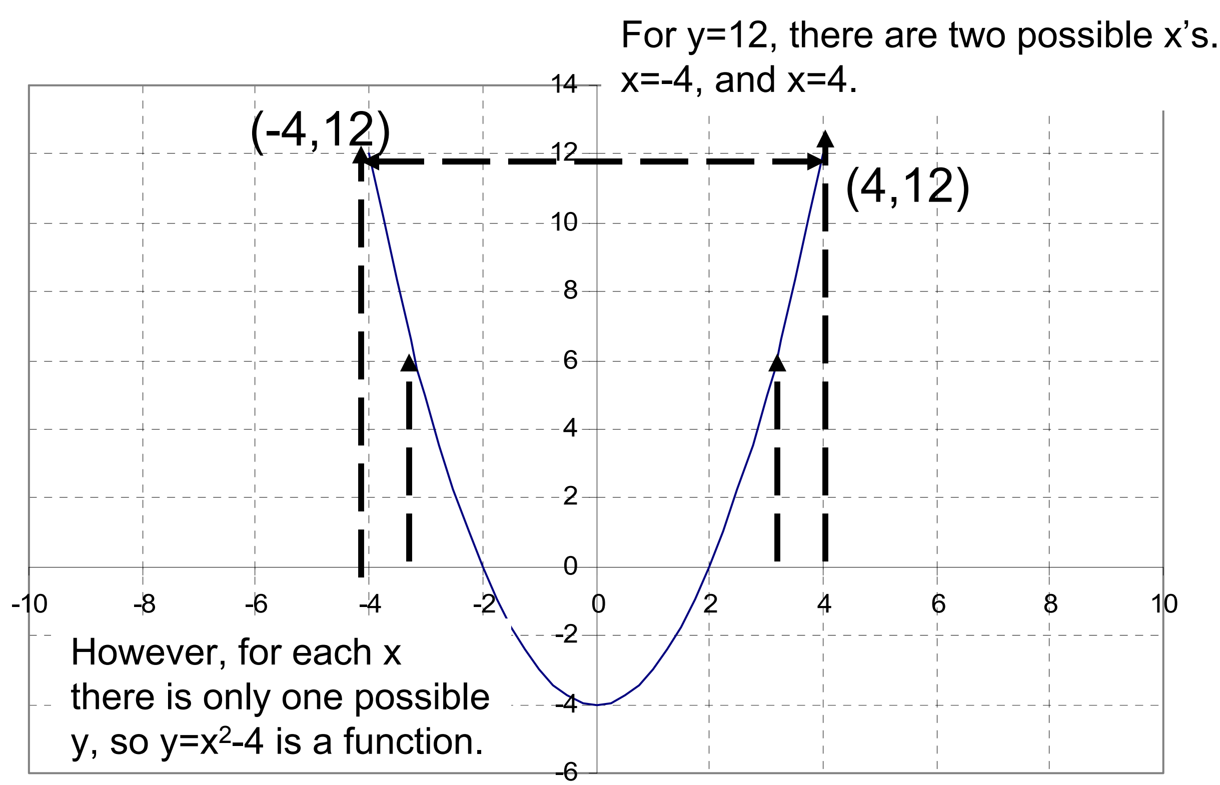
1.4 Relations and Functions

A relation is a correspondence between two sets. If x and y are two elements in these sets and if a relation exists between x and y, then ***x corresponds to y,*** or ***y depends on x***.

DEFINITION OF A FUNCTION:
Let X and Y two nonempty sets. A function from X into Y is a relation that associates with each element of X, ***<u>exactly one</u>*** element of Y.
However, an element of Y may have more than one elements of X associated with it.
That is for each ordered pair (x,y), there is exactly one y value for each x, but there may be multiple x values for each y. The variable x is called the **independent variable (**also sometimes called the **argument** of the function**)**, and the variable y is called **dependent variable (**also sometimes called the **image** of the function.)

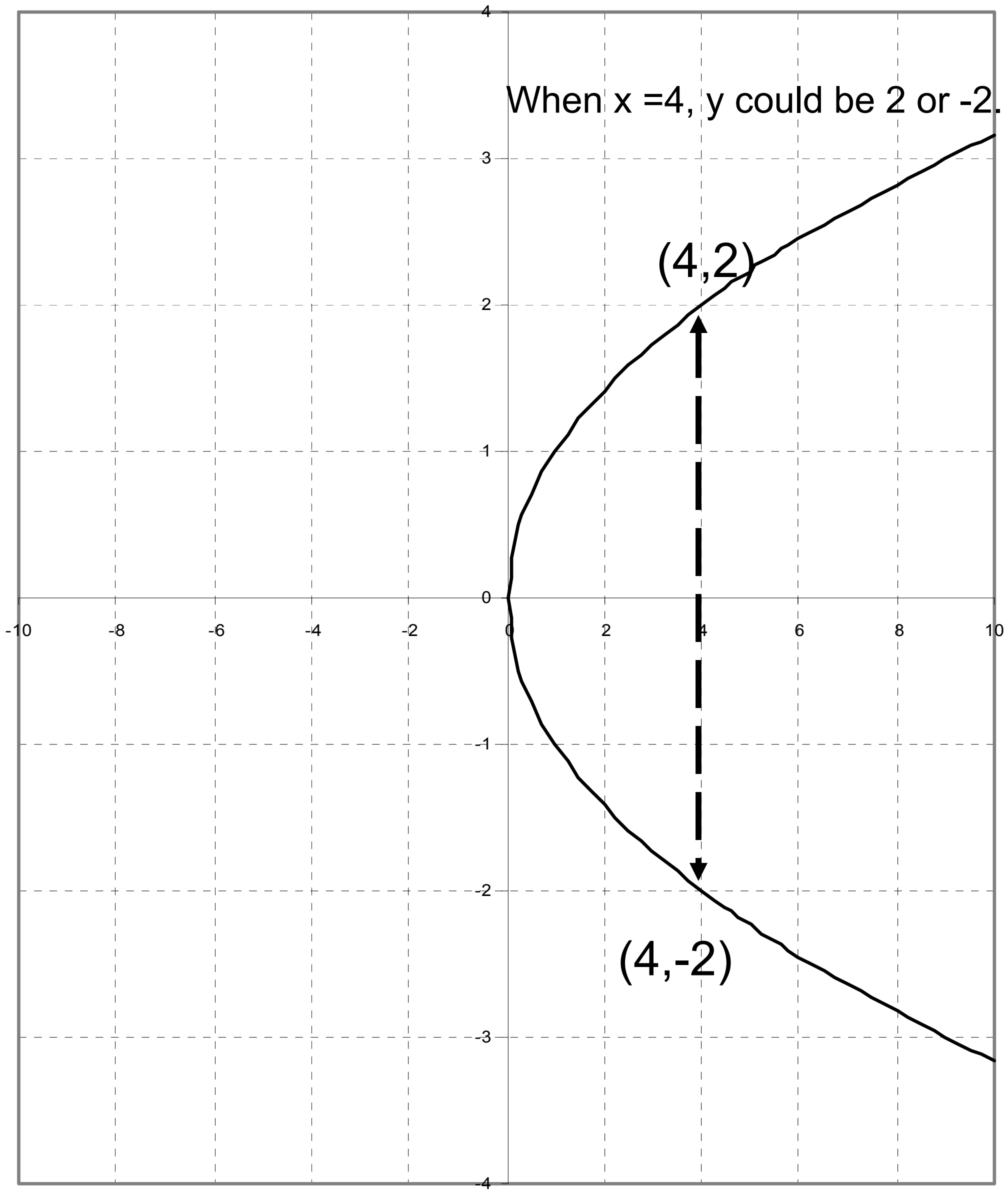
Below is the graph of $y=x^2-4$ (an upward parabola with vertex (0,-4))



<u>VERTICAL-LINE TEST THEOREM</u>
A set of points in the xy-plane is the graph of a function if and only if (*iff*), every vertical line intersects the graph in at most one point.

x = y² is not a function from X into Y, because there is not exactly one y value for each x.
Solving for y, you get y = $\pm\sqrt{x}$

which means there are two possible values for y.



This figure is a parabola with vertex at origin, and which axis of symmetry is with the x-axis, and opens to the right

Does this graph pass the vertical lines test?
Can you think of any other equations that are NOT functions of x?
A circle?

DOMAIN AND RANGE

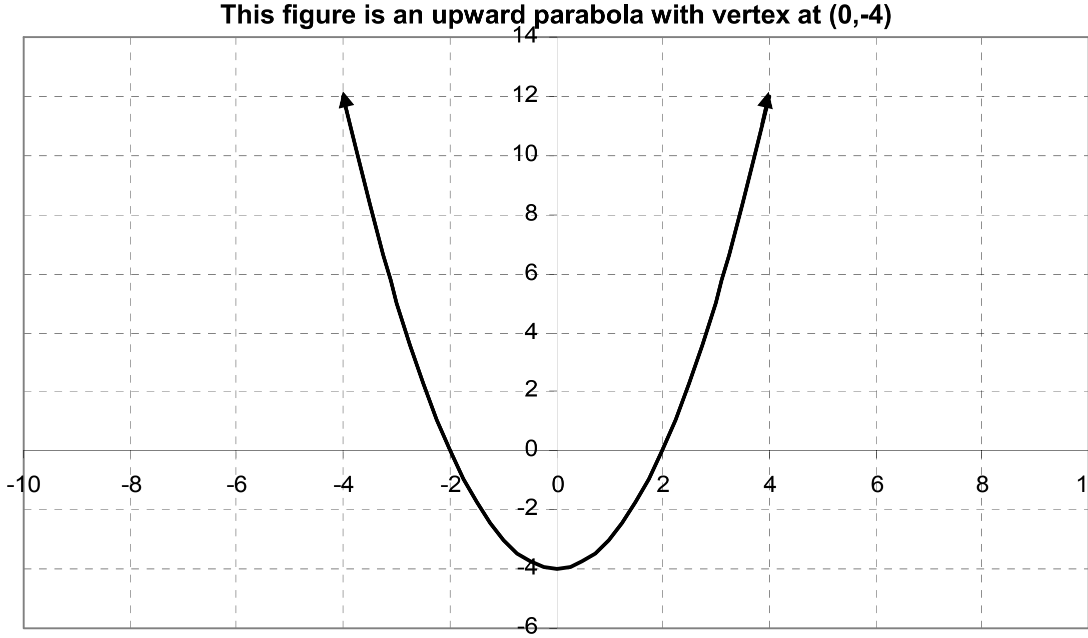
The set X is called the ***domain*** of the function. This is the set of all possible x values specified for a given function.

The set of all y values corresponding to X is called the ***range***.

In the example below, we see that x goes off into infinity in both directions, so the **domain** of $y=x^2$ is

{all real numbers}

However, we see there are no corresponding values of y that are less than -4, so the **range** is **{y | y≥-4}**



This figure is an upward parabola with vertex at (0,-4)

Example 4 p. 36

Consider the equation

$y = 2x - 5$, where the domain is $\{x|1 \le x \le 6\}$

Is this equation a function?

Notice that for any x, you can only get one answer for y.

(E.g. for x =1, y = 2(1) – 5= -3.) Therefore the equation is a function.

What is the range?

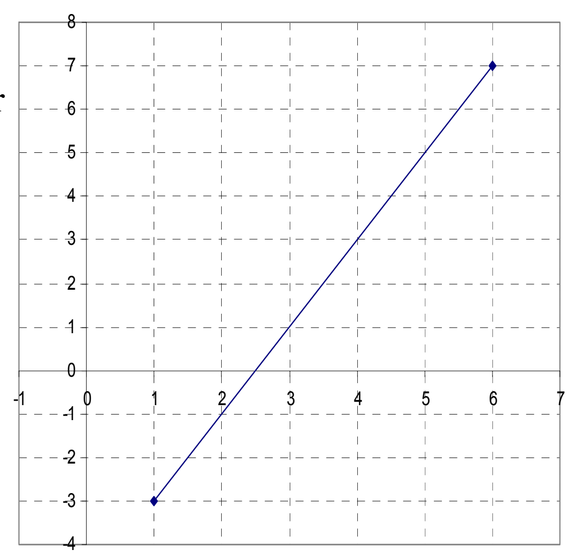
Since this is a straight line, we need only check y values at endpoints of domain. The y values are also called function values, so they are often referred to as f(x), which means ***the value of the function at x*** (not f times x).

The endpoints of the domain are 1 and 6.

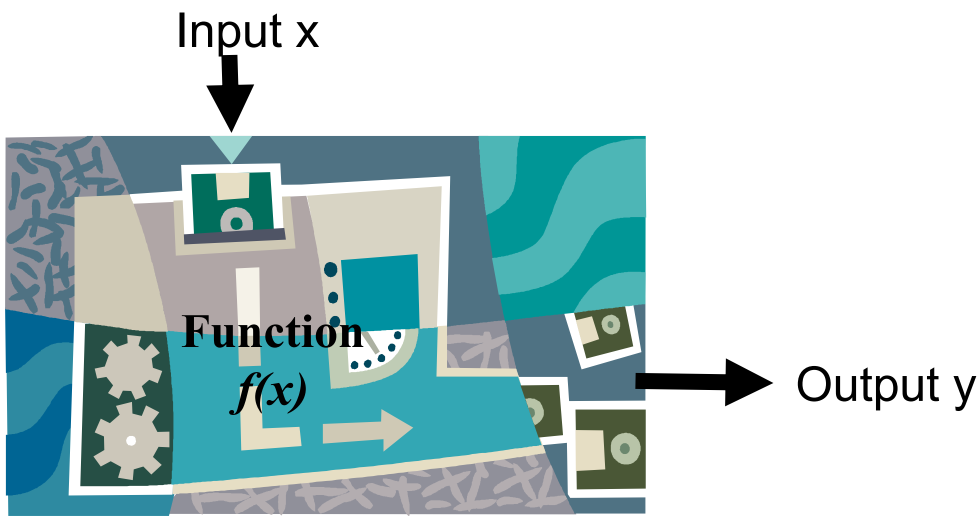
$f(1) = 2(1) - 5 = -3$

$f(6) = 2(6) - 5 = 7$

So the range is $\{y|-3 \le y \le 7\}$



This figure is a line segment with endpoints (1,-3) and (6,7).



A function, *f*, is like a machine that receives as input a number, x, from the domain, manipulates it, and outputs the value, y.

The function is simply the process that x goes through to become y. This "machine" has 2 restrictions:

1. It only accepts numbers from the domain of the function.
2. For each input, there is exactly one output (which may be repeated for different inputs).

Finding Values of a Function

Example 5 p. 38

For the function f defined by $f(x) = 2x^2 - 3x$, evaluate

b) $f(x) + f(3)$

$$= \overbrace{\left[2x^2 - 3x\right]}^{f(x)} + \overbrace{\left[2(3)^2 - 3(3)\right]}^{f(3)}$$

$$= 2x^2 - 3x + 18 - 9$$

$$\boxed{= 2x^2 - 3x + 9}$$

e) $f(x+3)$

$$= 2(x+3)^2 - 3(x+3)$$

$$= 2(x^2 + 6x + 9) - 3x - 9$$

$$= 2x^2 + 12x + 18 - 3x - 9$$

$$\boxed{= 2x^2 + 9x + 9}$$

Notice that $f(x) + f(3)$ does not equal $f(x+3)$

Difference Quotient of f

$$\frac{f(x+h)-f(x)}{h}=$$

$$=\frac{\left[2(x+h)^2-3(x+h)\right]-\left[2x^2-3x\right]}{h}$$

$$=\frac{\left[2(x^2+2hx+h^2)-3x-3h\right]-\left[2x^2-3x\right]}{h}$$

$$=\frac{2x^2+4hx+2h^2-3x-3h-2x^2+3x}{h}$$

$$=\frac{4hx+2h^2-3h}{h}$$

$$=\frac{h(4x+2h-3)}{h}$$

$$=4x+2h-3$$

This is called the ***difference quotient of f***, which is an important function in calculus. In calculus, the derivative, *dy/dx,* is defined as the limit of this function as h approaches 0.

IMPORTANT FACTS ABOUT FUNCTIONS

1. **For each *x* in the domain of a function *f*, there is one and only one image f(*x)* in the range.**
2. ***f*** **is the symbol that we use to denote the function. It is symbolic of the equation that we use to get from an *x* in the domain to the f(*x)* in the range.**
3. **If *y = f(x),* then *x* is called the <u>independent variable</u> or argument of *f*, and *y* is called the <u>dependent variable</u> or the value of f at *x* (or the image of f at *x*).**

Example 8 p. 40
Find the domain of each of the following functions:

$$\text{b) } g(x)=\frac{3x}{x^2-4} \qquad \text{c) } h(t)=\sqrt{4-3t}$$

The domain is the set of all possible x values that can be used in these functions.

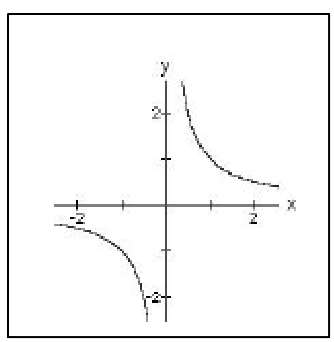
b) *g(x)* is the division of ***3x*** by ***x²-4***. This is undefined if the demoninator is 0, so we have the limitation that $x^2 - 4 \neq 0$.
Solve for x to find specifications for what x cannot be.
$x^2 \neq 4$
$x \neq \pm 2$

Therefore domain is ***{x|x≠ ±2}*** The function ***g(x)*** is **<u>not defined</u>** at ***x=2*** or ***x=-2.***

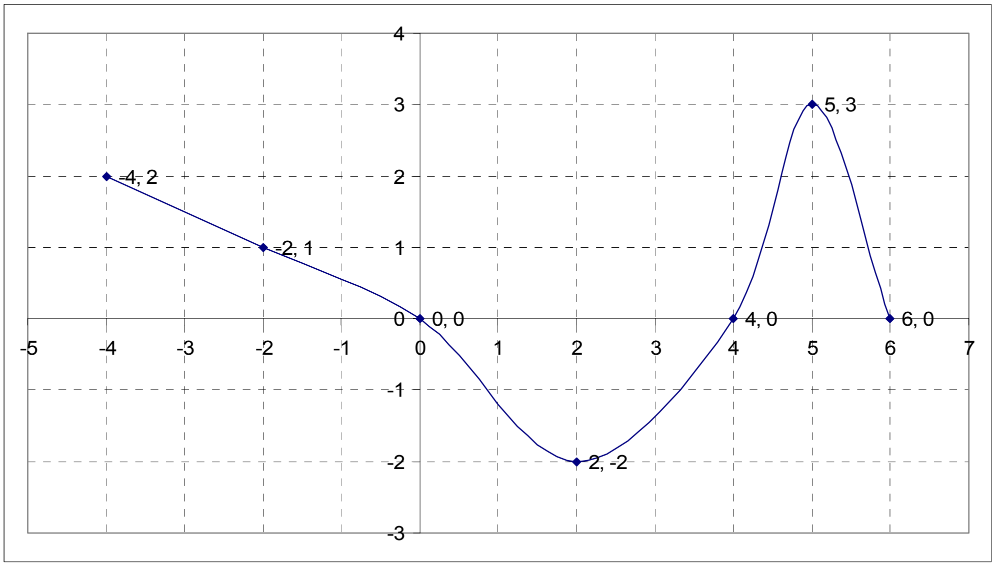
c) ***h(t)*** is the square root of *4-3t*. Only nonnegative numbers have real square roots, so the expression on the radical must be ≥ 0.
$4 - 3t \geq 0$
$-3t \geq -4$
Remember when you multiply an inequality by a negative number, the inequality reverses.
$-3t/(-3) \leq -4/(-3)$
$t \leq -4/3$
Therefore domain is ***{t|t ≤ -4/3}***
Another way to state this is in interval form: $\left(-\infty, \frac{-4}{3}\right]$ This is not a coordinate point. It's just another way to describe a set of numbers

The left-sided **(** means that *x* is open-bounded on the left by -infinity (of course it never reaches -infinity) and the right-sided **]** means that *x* is close-bounded on the right by -4/3. The set includes the number -4/3.

NOW YOU DO #37 on p.46
Look at the graph to the right (y=1/x):
Is this graph a function?
Yes, because a vertical line through any x-value on the graph only intersects the graph once.
What are the domain and range?
The domain (possible x values) is {x|x *{x|x≠ 0}*
The range (possible y values) is {y|y≠0}



Problem 48 on p. 47



a) Find *f(0)* and *f(6)*
What is y when x is 0 and x is 6? From the data given, we see the y-coordinate at x=0 is 0, so *f(0)=0.* The y-coordinate at x=6 is also 0, so *f(6)=0.*

b) Find *f(2)* and *f(-2)*
What is y when x is 2 and x is -2? From the data given, we see the y-coordinate at x=2 is -2, so *f(2)=-2.* The y-coordinate at x=-2 is 1, so *f(-2)=1.*

c) Is f(3) positive or negative? We see that at x=3 the graph is below the x-axis (where y < 0) so f(*3)* is negative.

d) Is f(*-1)* positive or negative? We see that at x=3 the graph is below the x-axis (where y < 0) so f(*3)* is negative.

e) For what numbers x is f(x) = 0? In other words, at which values of x cross the x-axis (where y=0)? The graph crosses the x-axis at x=0,x=4, x=6.

f) For what numbers x is f(x) <0? In other words, at which values of x is the graph below the x-axis? Remember, the coordinates where y=0 are not included. The graph is < 0 only for 0 < x < 4. In interval form this is $(0,4)$.

g) What is the domain of f? Domain is the possible x values. Remember that this graph does not continue into infinity on both sides. It is only define for the graph drawn. Therefore, can infer that the possible x values are $-4 \le x \le 6$, or [-4,6]

h) What is the range of f? The y values range from as low as -2 to as high as 3, so range is $\{y|\ -2 \le y \le 3\}$ or [-2,3].

i) What are the x-intercepts? The x-intercepts are found when y=0, which are the points {(0,0), (4,0),(6,0)}.

j) What is the y-intercept? By definition, this would not be a function if it crossed the y-axis (or any other vertical line) more than once. The only point that does this is (0,0).

k) How often does the line y=-1 intersect the graph? If we draw a horizontal line through y=-1, we'd see it intersects twice.

l) How often does he line x=1 intersect the graph? Three times.

m) For what value of x does f(x) = 3? Remember f(x) is the same as y. What is x when y=5? There's only one point on the graph that gives a y-value of 3. That is when x=5.

n) For what value of x does f(x)=-2? There's only one point on the graph that gives a y-value of -2. That is when x=2.

Example 11 on p. 44

$$f(x) = \frac{x}{x+2}$$

a) Is the point (1, ½) on the graph of *f*? Substitute 1 for x and ½ for f(x) and see if the statement is true.
Does ½ = 1/(1+2)? ½ ≠ ⅓ Therefore (1, ½) is not on the graph.

b) If x = -1, what is f(x)? f(-1) = -1/(-1+2) = -1/1 = -1
The point at x=-1 is (-1,-1).

c) If f(x) = 2, what is x? YOU DO THIS?

Example 12 on p.45 Area of a Circle

$$A(r)= \pi r^2$$

where r represents the radius of the circle. The domain is $\{r|r > 0\}$. Why?

NOW YOU DO #87 on p.50

HOMEWORK

p. 46 #9,17, 25, 29, 39, 45, 47,65, 73, 85